NATURAL COHEN-MACAULAYFICATION OF SIMPLICIAL AFFINE SEMIGROUP RINGS

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Abstract. Let K be a field, B a simplicial affine semigroup, and $C(B)$ the corresponding cone. We will present a decomposition of $K[B]$ into a direct sum of certain monomial ideals, which generalizes a construction by Hoa and Stückrad. We will use this decomposition to construct a semigroup $\tilde{B}$ with $B \subseteq \tilde{B} \subseteq C(B)$ such that $K[\tilde{B}]$ is Cohen-Macaulay with the property: $\tilde{B} \subseteq \hat{B}$ for every affine semigroup $\hat{B}$ with $B \subseteq \hat{B} \subseteq C(B)$ such that $K[\hat{B}]$ is Cohen-Macaulay.

1. Introduction

By an affine semigroup we mean a finitely generated submonoid of $(\mathbb{Z}^n, +)$ for some $n \in \mathbb{N}^+$. Let K be an arbitrary field and B an affine semigroup; as usual $K[B]$ denotes the affine semigroup ring associated to B, that is, the K-vector space with basis $\{t^b \mid b \in B\}$ and multiplication given by the K-bilinear extension of $t^a \cdot t^b = t^{a+b}$. Let X be a subset of $\mathbb{Q}^n$; we define $C(X) := \{\sum \lambda_i x_i \mid \lambda_i \in \mathbb{Q}_{\geq 0}, x_i \in X\}$ to be the cone spanned by X. In the following we will assume that B is a simplicial affine semigroup, that means, by definition, we assume that there are linearly independent elements $e_1, \ldots, e_d \in B$ with $C(\{e_1, \ldots, e_d\}) = C(B)$. By $A := \langle e_1, \ldots, e_d \rangle$ we denote the submonoid of B generated by $e_1, \ldots, e_d$, thus, $T := K[A]$ is a polynomial ring in d variables. Note that $\dim T = \dim K[B] = d$. In [HS03, Proposition 2.2], Hoa and Stückrad introduced in the homogeneous case a decomposition of $K[B]$ into a direct sum of monomial ideals. Generalizing this result, we will construct certain monomial ideals I_j in T and certain elements $h_j \in C(B) \cap \mathbb{Z}^n$ such that

$$K[B] \cong \bigoplus_{j=1}^{f} I_j(-h_j) \tag{1}$$

as $\mathbb{Z}^n$-graded T-modules (see Proposition 2.1). Note that the grading on T and on $K[B]$ is always given by $\deg t^a = a$. Some properties of the ring $K[B]$ can be characterized in terms of the semigroup B, for example, the Cohen-Macaulay or the Buchsbaum property; for more information we refer to [Hoc72, GSW76, Sta78, Tru83, HT86, GR02, Sch04, Mor07]. In view of [Sta78, Theorem 6.4] our decomposition (1) can be used to describe the Cohen-Macaulay property, namely, the ring $K[B]$ is Cohen-Macaulay if and only if every ideal I_j is equal to T. In Section 3 we will consider the affine semigroup $\tilde{B} = \langle e_1, \ldots, e_d, h_1, \ldots, h_f \rangle$ generated

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by the shifts which occur in the decomposition. We will show in Proposition 3.4

$$K[\tilde{B}] \cong \bigoplus_{j=1}^{f} T(-h_j)$$

as $\mathbb{Z}^n$-graded T-modules; meaning $K[\tilde{B}]$ can be deduced from our decomposition of $K[B]$ replacing I_j by T for all $j = 1, \dots, f$. This shows that the ring $K[\tilde{B}]$ is always Cohen-Macaulay. Let B_{sat} denote the saturation of B (see Section 3); by a result of Hochster [Hoc72] the ring $K[B_{sat}]$ is Cohen-Macaulay, since the semigroup B_{sat} is normal, see also [BG09, Theorem 6.4]. By construction $\tilde{B}$ and B_{sat} are affine semigroups in $C(B)$ containing B (see Lemma 3.2), thus, it is natural to ask:

Question 1.1. Is there a uniquely determined affine semigroup $\hat{B}$ with $B \subseteq \hat{B} \subseteq C(B)$ such that $K[\hat{B}]$ is Cohen-Macaulay, which is minimal among all affine semigroups with these properties?

This question has a positive answer, more explicitly, $\tilde{B}$ has exactly this property, see Theorem 3.5. This implies that $\tilde{B}$ is always contained in B_{sat}, in fact, the semigroup $\tilde{B}$ could be smaller than B_{sat}, see Example 3.3.

In Section 2 we will introduce the decomposition of $K[B]$, after this we we will study the Cohen-Macaulay property in Section 3. Finally, we compare our results to the results of Goto, Suzuki, and Watanabe, see Remark 3.7. For unspecified notation we refer to [BG09, Eis95].

2. Decomposition of simplicial affine semigroup rings

We define the set

$$B_A := \{x \in B \mid x - a \notin B \ \forall a \in A \setminus \{0\}\}.$$

By construction we have if $x \notin B_A$ then $x + y \notin B_A$ for all $x, y \in B$. Moreover, for all $x \in B$ there is an $m \in \mathbb{N}^+$ such that $mx \in A$, since $C(B) = C(\{e_1, \dots, e_d\}) = C(A)$ by assumption. This shows that B_A is finite. By $G(X)$ we denote the group generated by X, for $X \subseteq \mathbb{Z}^n$. For an element $x \in G(B)$ denote by $\lambda_1^x, \dots, \lambda_d^x$ the uniquely determined elements of $\mathbb{Q}$ such that $x = \sum_{i=1}^d \lambda_i^x e_i$. It follows that $x \notin B$ in case that $\lambda_i^x < 0$ for some i. Hence for every $x \in B$ we can consider the element $y = x - \sum_{j=1}^d n_j e_j \in B$ with $n_j \in \mathbb{N}$ such that $\sum_{j=1}^d n_j$ is maximal; we get $y \in B_A$. Thus, for all $x \in B$ there is an $y \in B_A$ such that $x = y + \sum_{j=1}^d n_j e_j$ for some $n_j \in \mathbb{N}$. We define $x \sim y$ if $x - y \in G(A)$, hence $\sim$ is an equivalence relation on $G(B)$. Clearly, every element in $G(B)$ is equivalent to an element in $G(B) \cap D$, where

$$D := \{x \in \mathbb{Q}^n \mid x = \sum\nolimits_{i=1}^d \lambda_i e_i, \lambda_i \in \mathbb{Q} \text{ and } 0 \le \lambda_i < 1 \text{ for all } i = 1, \dots, d\}$$

and for all $x, y \in G(B) \cap D$ with $x \neq y$ we have $x \not\sim y$, since $e_1, \dots, e_d$ are linearly independent. Hence the number of equivalence classes $f := \#(G(B) \cap D)$ in $G(B)$ is finite. Every element in B is by construction equivalent to an element in B_A. On the other hand for $x \in G(B)$ we have $x = y - z$ with $y, z \in B$ and again $mz \in A$ for some $m \in \mathbb{N}^+$. By this we get

$$x = y + (m-1)z - mz \sim y + (m-1)z \in B,$$

hence there are exactly f equivalence classes in B, $G(B)$, $G(B)\cap D$, and in B_A. By $\Gamma_1,\dots,\Gamma_f$ we will denote the equivalence classes on B_A. We define

$$h_j := \sum_{i=1}^{d} \min\{\lambda_i^x \mid x\in\Gamma_j\}\, e_i,$$

for $j=1,\dots,f$, hence $h_j\in C(B)$ by construction. Let $i\in\{1,\dots,d\}$; since $\lambda_i^x-\lambda_i^y\in\mathbb{Z}$ for all $x,y\in\Gamma_j$, we get for all $x\in\Gamma_j$ that

$$x-h_j=\sum_{i=1}^{d}\lambda_i^x e_i-\sum_{i=1}^{d}\min\{\lambda_i^y\mid y\in\Gamma_j\}\,e_i=\sum_{i=1}^{d} n_i e_i$$

for some $n_i\in\mathbb{N}$, hence $x-h_j\in A$, in particular $x\sim h_j$, and therefore $h_j\in C(B)\cap G(B)$. By construction $\tilde{\Gamma}_j:=\{t^{x-h_j}\mid x\in\Gamma_j\}$ is a subset of the polynomial ring $T=K[A]$, thus, $I_j:=\tilde{\Gamma}_jT$ are monomial ideals in T for $j=1,\dots,f$. In case that $d\geq 2$ we always have $\operatorname{ht} I_j\geq 2$ (height) since $\gcd\tilde{\Gamma}_j=1$ for all $j=1,\dots,f$. By this we obtain that all ideals I_j are equal to T in case that $d=1$. In the following we are interested in the canonical $\mathbb{Z}^n$-grading on T and on $K[B]$, which is given by $\deg t^a=a$. Note that our construction is a generalization of that of [HS03, Section 2] for the homogeneous case; moreover, the next proof is similar as the proof of [HS03, Proposition 2.2]. However, to keep things self contained we will prove it.

Proposition 2.1. *There is an isomorphism of $\mathbb{Z}^n$-graded T-modules:*

$$K[B]\cong\bigoplus_{j=1}^{f} I_j(-h_j).$$

Proof. Define

$$\psi:\bigoplus_{j=1}^{f} I_j(-h_j)\to K[B],$$

by

$$\psi(x_1,\dots,x_f)=\sum_{j=1}^{f} x_j t^{h_j}.$$

By construction ψ is well defined and preserves the canonical grading. Let $t^x\in K[B]$, that is, $x\in B$. By construction, there is an $y\in B_A$ such that $x=y+\sum_{i=1}^{d} n_ie_i$ for some $n_i\in\mathbb{N}$. We have $y\in\Gamma_j$ for some j, hence $t^{y-h_j}\in I_j$ and therefore

$$\psi(0,\dots,0,t^{\sum_{i=1}^{d} n_ie_i+y-h_j},0,\dots,0)=t^{\sum_{i=1}^{d} n_ie_i+y-h_j}t^{h_j}=t^x,$$

since $t^{\sum_{i=1}^{d} n_ie_i}\in T$. This shows that ψ is surjective. Let $x\in\ker\psi$; since ψ is homogeneous we may assume that x is also homogeneous; meaning $x=(\alpha_1t^{c_1},\dots,\alpha_ft^{c_f})$ for some $\alpha_j\in K$ and some $c_j\in A$, $j=1,\dots,f$. We get

$$\psi(x)=\sum_{j=1}^{f}\alpha_jt^{c_j+h_j}=0.$$

By construction $c_i+h_i\not\sim c_j+h_j$ for all $i\neq j$, hence $c_i+h_i\neq c_j+h_j$ for all $i\neq j$. This shows that $\alpha_j=0$ for all $j=1,\dots,f$ and therefore ψ is injective. $\square$

Example 2.2 ([GR02, Example 10])**.** The following example was given in [GR02] to study the relation between the Cohen-Macaulay and the Buchsbaum property. Consider the simplicial affine semigroup $B=\langle(2,0),(0,1),(3,1),(1,2)\rangle$, say $A=\langle(2,0),(0,1)\rangle$. We have

$$B_A=\{(0,0),(3,1),(1,2)\}.$$

By this we get $\Gamma_1 = \{(0,0)\}$ and $\Gamma_2 = \{(3,1),(1,2)\}$, thus, $h_1 = (0,0)$, $h_2 = (1,1)$ and therefore $\tilde{\Gamma}_1 = \{1\}$ and $\tilde{\Gamma}_2 = \{t^{(2,0)}, t^{(0,1)}\}$. By Proposition 2.1 it follows that

$$K[B] \cong T(-(0,0)) \,\oplus\, (t^{(2,0)}, t^{(0,1)})T(-(1,1))$$

as $\mathbb{Z}^2$-graded T-modules.

3. Natural Cohen-Macaulayfication

Since B is a simplicial affine semigroup we get that the cone $C(B)$ is pointed, that is, if $x, -x \in C(B)$ it follows that $x = 0$. Hence B is a positive affine semigroup, meaning, 0 is the only unit in B, thus, we can fix a positive grading on $K[B]$; see [BG09, Page 58,59]. Denote by $T_+ := (t^{e_1}, \ldots, t^{e_d})T$ the homogeneous maximal ideal of T and by $H^i_{T_+}(M)$ the i-th local cohomology module of a T-module M with respect to T_+. For a general treatment of the Cohen-Macaulay property and of local cohomology we refer to [BH98] and to [BS98]. The following Theorem is due to Stanley and shows that our canonical decomposition can be used to characterize the Cohen-Macaulay property of $K[B]$:

Theorem 3.1 ([Sta78, Theorem 6.4])**.** *The following assertions are equivalent:*

(1) The ring $K[B]$ is Cohen-Macaulay.
(2) There exists $\gamma_1, \ldots, \gamma_f \in B$ such that every element $x \in B$ has a representation of the form $x = \gamma_j + \sum_{i=1}^{d} n_i e_i$ for some γ_j and some $n_i \in \mathbb{N}$.
(3) We have $\#\Gamma_j = 1$ for all $j = 1, \ldots, f$.
(4) We have $I_j = T$ for all $j = 1 \ldots, f$.

Proof. The equivalence between (1) and (2) was proven in [Sta78, Theorem 6.4], provided that $B \subseteq \mathbb{N}^n$. Since $C(A) = C(B)$ it follows that $t^{e_1}, \ldots, t^{e_d}$ is a homogeneous system of parameters of $K[B]$. Thus, $K[B]$ is a Cohen-Macaulay ring if and only if $K[B]$ is a free T-module by [BG09, Proposition 6.3]. So, (4) $\Rightarrow$ (1) by Proposition 2.1. In case that I_j is a proper ideal for some j we get that $\dim T/I_j \le d-2$, since $\operatorname{ht} I_j \ge 2$ in this case. We have $H^i_{T_+}(T/I_j) \neq 0$ for some i with $0 \le i \le d-2$, and $H^i_{T_+}(T) = 0$ for every $i \neq d$, for example, by [Eis05, Proposition A1.16]. From the long exact sequence

$$\cdots \longrightarrow H^i_{T_+}(T) \longrightarrow H^i_{T_+}(T/I_j) \longrightarrow H^{i+1}_{T_+}(I_j) \longrightarrow H^{i+1}_{T_+}(T) \longrightarrow \cdots$$

we obtain $H^i_{T_+}(T/I_j) \cong H^{i+1}_{T_+}(I_j)$ for all i with $0 \le i \le d-2$. Hence $H^i_{T_+}(I_j) \neq 0$ for some i with $1 \le i \le d-1$, and therefore $H^i_{T_+}(K[B]) \neq 0$ for some i with $1 \le i \le d-1$ as well by Proposition 2.1 and the fact that local cohomology commutes with direct sums. Thus, $K[B]$ is not a free T-module by a similar argument, and therefore (1) and (4) are equivalent. The assertions (2) and (3) are equivalent as well. Moreover, by construction (3) $\Rightarrow$ (4). In case that $\#\Gamma_j \ge 2$ for some j we get for all $x \in \Gamma_j$ an $i \in \{1, \ldots, d\}$ such that $\lambda_i^{h_j} < \lambda_i^x$ and hence $t^{x-h_j} \neq 1$. This shows that I_j is a proper monomial ideal in T and we are done. □

Let us consider an affine semigroup $\hat{B}$ with $B \subseteq \hat{B} \subseteq C(B)$. The semigroup $\hat{B}$ is again simplicial, since $C(\hat{B}) = C(B) = C(\{e_1, \ldots, e_d\})$. In the following we are interested in the semigroup generated by the shifts which occur in the decomposition. We set

$$\tilde{B} := \langle e_1, \ldots, e_d, h_1, \ldots, h_f \rangle,$$

and we define the saturation B_{sat} of B by $B_{sat} := C(B) \cap G(B)$. Note that $K[B_{sat}]$ is always Cohen-Macaulay by [Hoc72, Theorem 1], since B_{sat} is normal; see also [BG09, Theorem 6.4].

Lemma 3.2. *We have $B \subseteq \tilde{B} \subseteq B_{sat}$.*

Proof. We get $h_1 \ldots, h_f \in C(B) \cap G(B)$ and therefore $\tilde{B} \subseteq C(B) \cap G(B) = B_{sat}$. Let $x \in B$; by construction there is an $y \in B_A$ such that $x = y + \sum_{i=1}^{d} n_i e_i$ for some $n_i \in \mathbb{N}$. We have $y \in \Gamma_j$ for some $j \in \{1, \ldots, f\}$. Moreover, $y - h_j \in A$ this implies $x = h_j + \sum_{i=1}^{d} n_i' e_i$ for some $n_i' \in \mathbb{N}$ and therefore $x \in \tilde{B}$, since $e_i \in \tilde{B}$. $\square$

This shows that $\tilde{B}$ is simplicial, since $B \subseteq \tilde{B} \subseteq C(B)$.

Example 3.3. Let us again consider the affine semigroup $B = \langle (2,0), (0,1), (3,1), (1,2) \rangle$ with $A = \langle (2,0), (0,1) \rangle$. We have $\tilde{B} = \langle (2,0), (0,1), (1,1) \rangle$ and $B_{sat} = \mathbb{N}^2$. Hence $B \subsetneqq \tilde{B} \subsetneqq B_{sat}$. Moreover, $K[B]$ is not Cohen-Macaulay by Theorem 3.1, but Buchsbaum by [GR02]. One can show that $\tilde{B}_A = \{(0,0), (1,1)\}$ and therefore

$$K[\tilde{B}] \cong T(-(0,0)) \oplus T(-(1,1))$$

as $\mathbb{Z}^2$-graded T-modules. It follows that the ring $K[\tilde{B}]$ is Cohen-Macaulay by Theorem 3.1.

In view of the above Example and Proposition 2.1 it is natural to ask the following:

Question. Is $K[\tilde{B}]$ always isomorphic to the direct sum of T shifted by h_j?

This would imply that $K[\tilde{B}]$ is always Cohen-Macaulay. The next Proposition will give a positive answer to this question:

Proposition 3.4. *We have*

$$K[\tilde{B}] \cong \bigoplus_{j=1}^{f} T(-h_j)$$

as $\mathbb{Z}^n$-graded T-modules.

Proof. By the comments in the beginning of Section 2 we get that the number of equivalence classes on $\tilde{B}_A$ is equal to that on B_A, since $G(B) = G(\tilde{B})$. By construction we have $h_k \not\sim h_l$ for all $k, l \in \{1, \ldots, f\}$ with $k \neq l$, hence for all $x' \in \tilde{B}$ there is a $j \in \{1, \ldots, f\}$ such that $x' \sim h_j$. We will show that if $x' \sim h_j$ for some $j \in \{1, \ldots, f\}$, then $x' - h_j \in A$, that is, $x' = h_j + \sum_{i=1}^{d} n_i e_i$ for some $n_i \in \mathbb{N}$ and therefore $\Gamma_1' = \{h_1\}, \ldots, \Gamma_f' = \{h_f\}$ are the equivalence classes on $\tilde{B}_A$ and we are done by Proposition 2.1 and construction. Let $x' \in \tilde{B}$, that is,

$$x' = \sum\nolimits_{t=1}^{d} n_t' e_t + \underbrace{\sum\nolimits_{t=1}^{f} n_t h_t}_{=x} = \sum\nolimits_{t=1}^{d} n_t' e_t + x,$$

for some $n_t', n_t \in \mathbb{N}$. We show by induction over $n := \sum_{t=1}^{f} n_t$ that for h_j with $h_j \sim x$ for some $j \in \{1, \ldots, f\}$ we have $x - h_j \in A$ and therefore $h_j \sim x'$ and $x' - h_j \in A$ as well.

<u>$n = 0$</u>: This is clear, since $h_j = 0$ for some $j \in \{1, \ldots, f\}$.

<u>$n > 0$</u>: We have $x = \sum_{t=1}^{f} n_t h_t = x'' + h_i$ for some $i \in \{1, \ldots, f\}$, by induction there is a h_j for some $j \in \{1, \ldots, f\}$ such that $h_j \sim x''$ and $x'' - h_j \in A$. We have $x \sim h_l$ for some $l \in \{1, \ldots, f\}$. It is now sufficient to show that $h_j + h_i - h_l \in A$; this implies $x - h_l \in A$, since $x'' - h_j \in A$. Note that $h_l \sim x'' + h_i \sim h_j + h_i$. We show that for $k = 1, \ldots, d$ we have $\lambda_k^{h_l} \leq \lambda_k^{h_j} + \lambda_k^{h_i}$. This implies

$$h_j + h_i - h_l = \sum\nolimits_{t=1}^{d} \underbrace{(\lambda_t^{h_j} + \lambda_t^{h_i} - \lambda_t^{h_l})}_{\geq 0} e_t \in A,$$

since $\lambda_k^{h_j} + \lambda_k^{h_i} - \lambda_k^{h_l} \in \mathbb{Z}$ for all $k = 1, \dots, d$. Let $\Gamma_1, \dots, \Gamma_f$ be the equivalence classes on B_A. Fix one $k \in \{1, \dots, d\}$. By construction there is an element $y_{jk} \in \Gamma_j$ with $\lambda_k^{y_{jk}} = \lambda_k^{h_j}$ and some $y_{ik} \in \Gamma_i$ with $\lambda_k^{y_{ik}} = \lambda_k^{h_i}$. Note that $y_{jk} \sim h_j$ and $y_{ik} \sim h_i$, hence

$$y_{jk} + y_{ik} \sim h_j + h_i \sim h_l. \tag{2}$$

We have $y_{jk} + y_{ik} \in B$ and therefore there is an $s \in B_A$ such that:

$$y_{jk} + y_{ik} = s + \sum\nolimits_{t=1}^{d} n_t e_t,$$

for some $n_t \in \mathbb{N}$. Clearly, $\lambda_k^s \leq \lambda_k^{y_{jk}} + \lambda_k^{y_{ik}} = \lambda_k^{h_j} + \lambda_k^{h_i}$. We have $h_l \overset{(2)}{\sim} y_{jk} + y_{ik} \sim s$ and therefore $\lambda_k^{h_l} \leq \lambda_k^s$, since $s \in \Gamma_l$. This implies $\lambda_k^{h_l} \leq \lambda_k^{h_j} + \lambda_k^{h_i}$ as required. $\square$

That means that $K[\tilde{B}]$ can be deduced from the decomposition of $K[B]$ in Proposition 2.1 replacing I_j by T for all $j = 1, \dots, f$. We will now give an answer to Question 1.1 raised in the introduction.

Theorem 3.5. *Let B be a simplicial affine semigroup, and $\tilde{B}$ be as above. The ring $K[\tilde{B}]$ is Cohen-Macaulay for the affine semigroup $\tilde{B}$ with $B \subseteq \tilde{B} \subseteq C(B)$, moreover, if $\hat{B}$ is an affine semigroup with $B \subseteq \hat{B} \subseteq C(B)$ such that the ring $K[\hat{B}]$ is Cohen-Macaulay, then $\tilde{B} \subseteq \hat{B}$.*

Proof. By Proposition 3.4 and Theorem 3.1 we get that the ring $K[\tilde{B}]$ is Cohen-Macaulay, moreover, by Lemma 3.2 $B \subseteq \tilde{B} \subseteq C(B)$. Let $\hat{B}$ be an affine semigroup with $B \subseteq \hat{B} \subseteq C(B)$ such that $K[\hat{B}]$ is Cohen-Macaulay; again $\hat{B}$ is simplicial. We show that $h_j \in \hat{B}$ for all $j = 1, \dots, f$ and therefore $\tilde{B} \subseteq \hat{B}$, since $e_1, \dots, e_d \in \hat{B}$. By Theorem 3.1 we know that $\hat{\Gamma}_1 = \{\hat{h}_1\}, \dots, \hat{\Gamma}_{\hat{f}} = \{\hat{h}_{\hat{f}}\}$ are the equivalence classes on $\hat{B}_A$. We have

$$f = \#(D \cap G(B)) \leq \#(D \cap G(\hat{B})) = \hat{f},$$

since $G(B) \subseteq G(\hat{B})$. For all $j \in \{1, \dots, f\}$ there is an $x \in B$ and an $i \in \{1, \dots, \hat{f}\}$ such that $x \sim h_j$ and $x \sim \hat{h}_i$, that is, we may assume $h_j \sim \hat{h}_j$ for $j = 1, \dots, f$. Fix one $j \in \{1, \dots, f\}$, and let $k \in \{1, \dots, d\}$; we will show that $\lambda_k^{\hat{h}_j} \leq \lambda_k^{h_j}$. This implies $\lambda_k^{h_j} - \lambda_k^{\hat{h}_j} \in \mathbb{N}$, since $\lambda_k^{h_j} - \lambda_k^{\hat{h}_j} \in \mathbb{Z}$. Thus, $h_j - \hat{h}_j \in A$ and therefore $h_j \in \hat{B}$, since $\hat{h}_j \in \hat{B}$ and $A \subseteq \hat{B}$. There is an element $x \in B$ with $x \sim h_j$ and $\lambda_k^x = \lambda_k^{h_j}$. Since $x \in \hat{B}$, $x \sim \hat{h}_j$, and $\hat{\Gamma}_j = \{\hat{h}_j\}$ we get $x = \hat{h}_j + \sum_{i=1}^d n_i e_i$ for some $n_i \in \mathbb{N}$ and therefore $\lambda_k^{\hat{h}_j} \leq \lambda_k^x = \lambda_k^{h_j}$. $\square$

Remark 3.6. There is an exact sequence of $\mathbb{Z}^n$-graded T-modules:

$$0 \longrightarrow K[B] \longrightarrow K[\tilde{B}] \longrightarrow K[\tilde{B} \setminus B] \longrightarrow 0.$$

By Proposition 2.1 and Proposition 3.4 we have

$$K[\tilde{B} \setminus B] \cong \bigoplus_{j=1}^{f} T/I_j(-h_j)$$

as $\mathbb{Z}^n$-graded T-modules. Hence $\dim K[\tilde{B} \setminus B] \leq \dim K[B] - 2$, since $\operatorname{ht} I_j \geq 2$; provided that $d \geq 2$.

Remark 3.7. Assume that $B \subseteq \mathbb{N}^n$ for some $n \in \mathbb{N}^+$. Let B' be the extension of B in $C(B)$ studied by Goto, Suzuki, and Watanabe in [GSW76], or by Hoa and Trung in [HT86] (see also [Mor07, Sch04]). They proved that $B = B'$ if and only if $K[B]$ is a Cohen-Macaulay ring. Since $B' = (B')'$, we get that $K[B']$ is Cohen-Macaulay, hence $\tilde{B} \subseteq B'$ by Theorem 3.5. Conversely, $\tilde{B} = \tilde{B}'$, since $K[\tilde{B}]$ is Cohen-Macaulay. We have $B \subseteq \tilde{B}$, hence $B' \subseteq \tilde{B}' = \tilde{B}$ and therefore $\tilde{B} = B'$.

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